The Qualitative Expectations Hypothesis: Model Ambiguity, Consistent Representations of Market Forecasts, and Sentiment

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Abstract

We introduce the *Qualitative Expectations Hypothesis* (QEH) as a new approach to modeling macroeconomic and Financial outcomes. Building on John Muth's seminal insight underpinning the Rational Expectations Hypothesis (REH), QEH represents the market's forecasts to be consistent with the predictions of an economist's model. However, by assuming that outcomes

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lie within stochastic intervals, QEH, unlike REH, recognizes the ambiguity faced by an economist and market participants alike. Moreover, QEH leaves the model open to ambiguity by not specifying a mechanism determining specific values that outcomes take within these intervals. In order to examine a QEH model's empirical relevance, we formulate and estimate its statistical analog based on simulated data. We show that the proposed statistical model adequately represents an illustrative sample from the QEH model. We also illustrate how estimates of the statistical model's parameters can be used to assess the QEH model's qualitative implications.

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1 Introduction

We introduce the *Qualitative Expectations Hypothesis* (QEH) as a new approach to modeling macroeconomic and financial outcomes. QEH recognizes that economists and market participants alike face ambiguity about which is the correct quantitative model of the process driving outcomes. Building on Frank Knight's distinction between risk and "true uncertainty," QEH formalizes ambiguity by opening an economic model to unforeseeable change in its coefficients. The defining feature of such change is that it cannot "by any method be [represented ex ante] with an objective, quantitatively determined probability" (Knight, 1921, p. 321). Hence, we do not impose a probabilistic structure on the change in the model's coefficients over time – and thus we do not specify a complete dynamic stochastic process driving outcomes. Instead, we assume that the unforeseeable change in the coefficients is moderate.[1] We formalize this change by restricting it to stochastic intervals driven by the evolution of the fundamental variables in the model. As a result, the model assumes that outcomes lie within stochastic intervals. But we leave the model open to ambiguity by not specifying a mechanism determining specific values that the outcomes can take within these intervals.

By definition, when an economist formulates his model, he hypothesizes how outcomes unfold over time. John Muth's (1961, p. 315) seminal insight was that representations of the market's forecast that deviate systematically from the predictions of the model contradict the economist's hypothesis. Consequently, Muth proposed that it would be "sensible" for the economist to represent the market's forecast as being consistent with the process assumed by his model to be driving outcomes.

Building on Muth's seminal insight, a QEH model represents the market's forecasts to be consistent with the assumption that an economist and market participants face ambiguity. To this end, we introduce a conditional qualitative expectation (QE) of future outcomes, which we define as the conditional expectation of the upper and lower bounds of these outcomes' assumed stochastic intervals. The QE measures the intervals within which future outcomes, according to the model, are expected to lie.

Representing the market's forecasts to lie within these intervals, but stopping short of specifying a mechanism determining the particular values that these forecasts take, is the key feature that distinguishes QEH from the Rational Expectations Hypothesis (REH). While both approaches rely on model consistency, REH, unlike QEH, represents the market's forecast on the basis of a model that rules out unforeseeable change and thus ambiguity about which representation of outcomes is correct. An REH model does so by specifying all changes

[1] This assumption underpins the Imperfect Knowledge Economics (IKE) approach to macroeconomics and finance theory (Frydman and Goldberg, 2007, 2011).

in variables and coefficients with a probabilistic structure, thereby assuming that outcomes follow a dynamic stochastic process. The model represents the market's forecast with the conditional expectation of this process. Given the realizations of the fundamental variables, this REH representation determines a particular value for the market's forecast.

Applying REH models to the study of movements in asset prices and risk has revealed many empirical puzzles.[2] Frydman and Goldberg (2011, 2013) trace these anomalies to REH models' premise that unforeseeable change is unimportant for understanding asset prices and risk. Lars Hansen (2013, p. 399) conjectures that these puzzles arise from REH's narrow representation of uncertainty as "risk conditioned on a model." He points out that REH representations "miss something essential: uncertainty [arising from] ambiguity about which is the correct model."

REH models' empirical difficulties gave rise to behavioral finance, which has sought to remedy these difficulties by relating the market's forecast to psychological factors, such as market sentiment – its optimism or pessimism about the future course of prices.[3] However, because behavioral-finance models assume away unforeseeable change, they represent the market's forecast in ways that are inconsistent with the stochastic process assumed by an economist to be driving outcomes.

By recognizing ambiguity, a QEH model can account for the role of both the fundamental factors on which REH models focus and the psychological factors underpinning behavioral-finance models. And it can do so without abandoning model consistency.

Because a QEH model assumes that market forecasts lie within the stochastic intervals that are consistent with the process an economist assumes is driving outcomes, there are myriad possible model-consistent quantitative forecasts. In making decisions – for example, about how many stocks to buy or sell – market participants thus face inherent ambiguity. They select particular quantitative forecasts by relying on a combination of considerations, including formal (econometric) models, market sentiment, and other non-fundamental factors. A QEH model can formalize the qualitative effect of such factors on participants' model-consistent forecasts, by imposing additional restrictions on how the market, in forming these forecasts, revises the weights it attaches to fundamentals.

We use a simple stock-price model to illustrate how QEH represents how the market relates outcomes to fundamentals, and how psychological factors might affect model-consistent forecasts. The model rests on a no-arbitrage condition, which assumes that at any point in time, market participants bid the price to the level equal to the market's forecast of the

[2] Hansen (2013) provides an extensive discussion of econometric studies that have uncovered these anomalies.

[3] For surveys of the behavioral-finance approach see Barberis and Thaler (2003), Shleifer (2000), and references therein.

next-period discounted sum of the dividends and price.

QEH represents this forecast as being consistent with the model's qualitative expectations of dividends and prices. To this end, we assume that corporate earnings drive dividends, and we formalize ambiguity by opening the model to moderate unforeseeable change in the coefficient determining the impact of earnings on dividends. According to the model, future dividends and prices are expected to lie within the QE intervals of these outcomes. Conditional on current earnings, QEH's model-consistent representations of the market's forecasts of future dividends and prices lie within these intervals.

With these representations, we derive implications of the no-arbitrage condition. Under REH, such a condition implies that, given the value of earnings at a point in time, the model determines the precise value of the price set by the market. Recognizing ambiguity changes the implications of a standard no-arbitrage condition substantially. Because a QEH model does not determine the precise values of the market's forecasts of dividends and prices, it does not determine the precise value of the price the market sets.

We show that our QEH model represents an asset price to lie within an interval. However, despite implying such ambiguity regarding the precise price, the model relates the stock price, p_t, to corporate earnings, x_t, in a way that is consistent with the representations of the dividend process as well as with the no-arbitrage condition. That is, $p_t = \theta_t x_t$, where the coefficient $\theta_t > 0$ lies within a time-varying interval determined by the discounted expected intervals for future dividends. Moreover, we relate θ_t to the coefficients in the model's representation of the market's forecasts of next-period dividends and prices.

We show that, in general, the model-consistent representation of the market's forecasts does not imply a positive co-movement between the price and earnings, defined as

$$\Delta p_t / \Delta x_t \geq 0, \tag{1}$$

where $\Delta p_t = p_t - p_{t-1}$. The reason is that changes in the impact of earnings on the forecasts of future dividends and prices, which determine the change in the coefficient θ_t, can outweigh the effect of the change in earnings on the price. As a result, the price can decrease when earnings increase. However, under the additional condition that θ_t changes moderately with x_t, the price $p_t = \theta_t x_t$ co-moves positively with earnings.

We illustrate how one can formalize the qualitative effect of psychological factors by introducing a market sentiment index, i_t, which can be positive or negative.[4] Specifically,

[4] A large literature has emerged from the construction of proxies for investor sentiment based on narrative reporting in the Wall Street Journal, Bloomberg News, and elsewhere. Using such proxies, a number of empirical studies have shown that market sentiment has a significant effect on stock-price movements. For such studies of the U.S. stock market, see, for example, Baker and Wugler (2006, 2007), Tetlock (2007), Garcia (2013), and Mangee (2017a,b). Recently, Shiller (2017) strongly emphasized the importance of

we restrict the interval for the model-consistent representation of the market's forecasts of future dividends and prices – and thereby θ_t – to depend on i_t. We show that under the stated condition, stock-price movements are driven primarily by company earnings, with market sentiment playing an amplifying role.

Because the QEH model does not specify a complete stochastic process for outcomes, it cannot be directly estimated on the basis of time-series data. Thus, the model's implications concerning co-movements in these data cannot be directly tested. In order to assess the empirical adequacy of the QEH model's assumptions and its qualitative implications, we propose a statistical model for dividends and earnings. This statistical analog represents the QEH model's moderately changing coefficient of earnings on dividends with a time-varying random coefficient.

A class of general autoregressive score (GAS) models seems particularly suitable to serve as the basis for formulating statistical representations of QEH-based models.[5] Building on the GAS approach, we formulate a statistical model that captures the QEH model's dynamics of earnings and dividends. We also propose a way to simulate earnings, dividends and prices in a way that is consistent with the QEH model.

We leave the development of econometric methodology for QEH models for future research. However, we use simulated data to illustrate how estimates of the GAS analog can be used to assess a QEH model's empirical adequacy. To this end, we estimate the GAS model with maximum likelihood, and rely on standard misspecification tests to assess the model's adequacy in representing the data. We show that the misspecification tests are not rejected for the residuals of the GAS model. Thus, in the context of our illustration, the GAS model's simple, parsimonious structure provides an adequate representation of the data simulated from the highly non-linear QEH model.[6]

We use the GAS model's estimates to construct simple summary measures designed to assess the adequacy of a QEH model's assumptions and implications – for example, that the structure of the dividend process changes moderately, or that the price lies within a model-consistent interval. We show that these assumptions and implications seem to hold in the sample considered in our illustration.

The structure of the paper is as follows: In Section 2, we formulate the components of the model and formalize ambiguity by opening the model to unforeseeable change. Section 3 defines the concept of the model's qualitative expectation (QE), which serves as the basis of the Qualitative Expectations Hypothesis (QEH) formulated in Section 4. Section 4 also

narrative accounts for informing econometric modeling of asset markets.

[5] For the development of GAS models, see Blasques *et al* (2014b), Harvey and Luati (2014), Koopman *et al* (2016), and references therein.

[6] The importance of achieving a well-specified model for reliable inference is emphasized by Hendry (1995).

shows how QEH represents the model-consistent market forecasts of dividends and prices under ambiguity. These representations are used in Section 5 to define the no-arbitrage interval condition, which is the QEH counterpart to the standard REH-based no-arbitrage condition. In Section 6, we show that the QEH-based no-arbitrage condition implies that the market price lies within an interval and that it is related to earnings in a way that varies over time. In Section 7, we establish conditions under which prices and earnings co-move positively. In Section 8, we formalize a qualitative effect of market sentiment on co-movements between prices and earnings, by imposing additional restrictions on the intervals for the market forecasts. In Section 9, we sketch our proposed approach to the econometric analysis of QEH models. We formulate a GAS analog to the QEH model in Section 2, and estimate it on the basis of simulated data. Using the GAS analog's estimates, we illustrate how the adequacy of QEH model's assumptions and qualitative implications can be assessed empirically. Section 10 concludes the paper with remarks about the recasting suggested by QEH of the role of psychological considerations in rational forecasting, and about the QEH model's potential to shed new light on one of the long-standing puzzles in financial economics – the role of fundamental and psychological considerations in driving long swings in asset prices.

2 Opening an Asset-Pricing Model to Unforeseeable Change and Ambiguity

We illustrate QEH in the context of a simple stock-pricing model in which we introduce ambiguity by allowing for unforeseeable change in its coefficients. The model rests on an assumption that summarizes how the market sets the stock price at each point in time. Participants bid the price to the level that satisfies the following no-arbitrage condition:

$$p_t = \gamma \left[\mathcal{F}_t \left(d_{t+1} \right) + \mathcal{F}_t \left(p_{t+1} \right) \right] \quad \text{for } t = 1, 2, 3, ... \tag{2}$$

where p_t is the market price, d_t denotes dividends, $\mathcal{F}_t(\cdot)$ stands for the time-t values of the market's (an aggregate of its participants') forecasts of dividends and prices at time $t+1$, and γ is a discount factor, which, for simplicity, we set equal to a constant.

In order to derive the testable implications of the no-arbitrage condition in (2) we must represent the values of the market's forecasts formally. As we discuss in Section 4, QEH does so by recognizing the ambiguity faced by an economist and market participants. It represents the market's forecasts of dividends and prices as being consistent with the model's assumptions about the processes underpinning these outcomes. To this end, we consider a simple model for the dividend process, which relates dividends to one fundamental factor, corporate earnings, which we denote by x_t:

$$d_t = b_t x_t + \varepsilon_{dt}, \tag{3}$$

where ε_{dt} are i.i.d.$(0, \sigma_d^2)$ innovations and b_t is the time-varying impact of earnings on dividends. We assume log-earnings follow a random walk with drift,

$$\Delta \log x_t = \mu + \varepsilon_{xt}, \tag{4}$$

where ε_{xt} are i.i.d.$(0, \sigma_x^2)$. Finally, we condition on the initial value $\log x_0$, and we choose the drift μ, so that x_t is a martingale, $E(x_t|x_{t-1}) = x_{t-1}$. See Appendix A for the choice of μ for Gaussian errors.

Next, we formalize ambiguity by opening the model to unforeseeable change in the impact coefficient, b_t, of earnings on dividends in (3). In general, we could do so in any part of the model. Here, we specify a stochastic process for earnings, but formalize ambiguity about the dividend process by stopping short of imposing a probabilistic structure on the sequence of coefficients b_t.

However, in order for the model in (3) to serve as the basis for representing the market's forecast, we must constrain *ex ante* how b_t unfolds over time. We hypothesize that b_t is positive at every point in time,

$$b_t > 0. \tag{5}$$

We also specify a rule limiting changes in b_t by formalizing the distinction between moderate and non-moderate change in the dividend process. Building on Frydman and Goldberg (2007), we define the change in this process to be moderate if it can be represented by limiting the change $\Delta b_{t+1} = b_{t+1} - b_t$ as follows,

$$\frac{|\Delta b_{t+1}|}{b_t} \leq \frac{|\Delta x_{t+1}|}{x_{t+1}}. \tag{6}$$

This moderate change (MC) condition constrains the change in b_t between the adjacent points in time to be contingent on the realized change in x_t. Because earnings and b_t are both positive, (6) may be stated equivalently as the interval for b_{t+1} given b_t, x_t, and x_{t+1},

$$b_{t+1} \in \left[b_t \left(1 - \frac{|\Delta x_{t+1}|}{x_{t+1}} \right)^+, b_t \left(1 + \frac{|\Delta x_{t+1}|}{x_{t+1}} \right) \right], \tag{7}$$

where $a^+ = \max(0, a)$. Thus, at time $t + 1$, the value of b_{t+1} is assumed to lie in a random interval as defined by the previous value b_t and by x_t and x_{t+1}. Importantly, there is no rule or mechanism that determines the value of b_t within this interval.

Multiplying in (7) by x_{t+1}, it follows that

$$b_{t+1}x_{t+1} \in b_t\mathcal{I}_{t+1}, \text{ where } \mathcal{I}_{t+1} = [(x_{t+1} - |\Delta x_{t+1}|)^+, (x_{t+1} + |\Delta x_{t+1}|)], \tag{8}$$

or equivalently, $b_{t+1} \in b_t\mathcal{I}_{t+1}/x_{t+1}$, such that for dividends we find that

$$d_{t+1} \in \mathcal{I}^d_{t+1} = [b_t(x_{t+1} - |\Delta x_{t+1}|)^+ + \varepsilon_{dt+1}, b_t\left(x_{t+1} + |\Delta x_{t+1}|\right) + \varepsilon_{dt+1}] = b_t\mathcal{I}_{t+1} + \varepsilon_{dt+1}. \tag{9}$$

This illustrates a key consequence of recognizing ambiguity about the dividend process: the model states that dividends lie within random intervals that vary over time, but it does not specify a mechanism determining the values of dividends within each interval. We refer to such outcomes as *qualitative implications* of the QEH model.

Beyond yielding qualitative implications about the values of outcomes at a point in time, a QEH model also generates qualitative predictions about the co-movements in time-series data. The following Lemma states that the MC condition implies positive co-movement between expected dividends and earnings.[7]

Lemma 1 *If the change in b_t satisfies the MC condition in (6), expected dividends, $E(d_t|x_t) = b_tx_t$, co-move positively with earnings, x_t,*

$$\Delta E(d_t|\mathbf{x}_t)\Delta x_t \geq 0. \tag{10}$$

Proof of the lemma is given in Appendix A.

To complete the QEH model, we need to represent the market's forecasts in the no-arbitrage condition in (2). To this end, we next define the concept of qualitative expectations. We represent the market's forecast in a way that is consistent with the model's qualitative expectations. We then show that under QEH, the stock price lies within the model-consistent interval.

3 The Model's Qualitative Expectations

We first consider the expectation of dividends one period ahead. Conditional on the time-t information on earnings, $\mathbf{x}_t$, our representation for the dividend process in (3) implies that

$$E\left(d_{t+1}|\mathbf{x}_t\right) = E\left(b_{t+1}x_{t+1} + \varepsilon_{dt+1}|\mathbf{x}_t\right) = E\left(b_{t+1}x_{t+1}|\mathbf{x}_t\right).$$

[7]See (1), and the further discussion in Section 7.

Recall that the QEH model assumes a probabilistic structure on x_{t+1}, while there is no rule for determining b_t other the condition than it belongs to the interval (7). Consequently, we cannot evaluate $E\left(b_{t+1}x_{t+1}|\mathbf{x}_t\right)$, which implies that we cannot compute the conditional expectation of future dividends. However, conditional on $\mathbf{x}_t$, we can compute the expected interval for d_{t+1} as the conditional expectation of the upper and lower limits in (9). We formalize this by defining the conditional qualitative expectations (QE) as follows:

Definition 1 (Conditional Qualitative Expectation)

For any random interval defined by $[X_L, X_U]$, where $X_L \leq X_U$ are random variables with finite expectation, the conditional Qualitative Expectation, $QE_t(\cdot)$, given available information $\mathbf{x}_t = (x_1, ..., x_t)$, is defined as follows:

$$QE_t\left([X_L, X_U]\right) = \left[E\left(X_L|\mathbf{x}_t\right), E\left(X_U|\mathbf{x}_t\right)\right].$$

We collect some properties of the conditional qualitative expectations in the next lemma.

Lemma 2 *For random intervals X and Y, $\lambda \in R$ and a stochastic process x_t,*

(i)	*Additivity*	$QE_t\left(X+Y\right) = QE_t\left(X\right) + QE_t\left(Y\right),$
(ii)	*Homogeneity*	$QE_t\left(\lambda X\right) = \lambda QE_t\left(X\right),$
(iii)	*Conditional Expectation Consistency*	$QE_t\left(x_{t+1}\right) = E\left(x_{t+1}|\mathbf{x}_t\right).$
(iv)	*Iterated QE*	$QE_t\left([X_L, X_U]\right) = QE_t\left(QE_{t+1}\left([X_L, X_U]\right)\right)$

We now use the $QE_t(\cdot)$ to derive the expected intervals for future dividends. From (9), $d_{t+1} \in \mathcal{I}^d_{t+1} = b_t\mathcal{I}_{t+1} + \varepsilon_{dt+1}$, implies that $QE_t\left(d_{t+1}\right) \in QE_t\left(\mathcal{I}^d_{t+1}\right)$, and therefore

$$\begin{aligned} QE_t\left(\mathcal{I}^d_{t+1}\right) &= QE_t\left(b_t\mathcal{I}_{t+1} + \varepsilon_{dt+1}\right) \\ &= \left[E\left(b_t\left(x_{t+1} - |\Delta x_{t+1}|\right)^+ + \varepsilon_{dt+1}|\mathbf{x}_t\right), E\left(b_t\left(x_{t+1} + |\Delta x_{t+1}|\right) + \varepsilon_{dt+1}|\mathbf{x}_t\right)\right] \\ &= \left[b_tE\left(\left(x_{t+1} - |\Delta x_{t+1}|\right)^+|\mathbf{x}_t\right), b_tE\left(x_{t+1} + |\Delta x_{t+1}||\mathbf{x}_t\right)\right] \\ &= b_tx_t\left[L, U\right], \end{aligned} \tag{11}$$

where the conditional expectations L and U depend on the distribution of ε_{xt}. In Appendix B, values for L and U are derived for the case of Gaussian innovations ε_{xt} in (4).

With the k-period iterated qualitative expectations given by,

$$QE^{(k)}_t\left(\cdot\right) = QE_t\left(QE_{t+1}\left(\cdots QE_{t+k}\left(\cdot\right)\cdots\right)\right), \; k = 0, 1, \ldots$$

we find that,

$$\begin{aligned} QE_t^{(1)}(d_{t+2}) &= QE_t(QE_{t+1}(d_{t+2})) \in QE_t\left(QE_{t+1}\left(\mathcal{I}_{t+2}^d\right)\right) \\ &= QE_t\left(b_{t+1}x_{t+1}[L,U]\right) = QE_t\left(b_{t+1}x_{t+1}\right)[L,U] \\ &\subseteq b_t x_t\left[L^2,U^2\right]. \end{aligned}$$

Furthermore, Lemma 2 (iv) and (9) for d_{t+2} imply that,

$$QE_t(d_{t+2}) = QE_t^{(1)}(d_{t+2}) \in b_t x_t\left[L^2,U^2\right]$$

In general, iterating k times,

$$QE_t(d_{t+k}) = QE_t^{(k-1)}(d_{t+k}) \in b_t x_t\left[L^k,U^k\right]. \tag{12}$$

4 The Qualitative Expectations Hypothesis

The no-arbitrage condition in (2) is a purely descriptive summary of how the market sets the price at each point in time, in the sense that it has no testable implications for how prices are determined at the point in time or for how they unfold over time. For example, in order to derive the implication of (2) for the relationship between prices and corporate earnings, an economist must formally relate $\mathcal{F}_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1})$ to these earnings. Importantly, as we show in Section 6.1, the assumed properties of such representations play a key role in deriving the model's predictions for time-series data.

We build on John Muth's fundamental insight to represent $\mathcal{F}_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1})$. According to Muth (1961, 315), given that an economist's model formalizes his hypothesis about how market outcomes will *actually* unfold over time, it would be "sensible" for him to represent the market's forecasts as being consistent with the predictions of his own model.

Recognizing ambiguity, however, requires jettisoning the assumption that change can be represented with a probabilistic rule and that a single conditional distribution represents how future outcomes actually unfold over time. Consequently, representing forecasts by participants who face ambiguity as consistent with the predictions of an economist's model requires that the model does *not* generate unique, precise predictions of market outcomes.

The following summarizes the key features of a QEH model and its model-consistent representation of the market's forecast.

The Qualitative Expectations Hypothesis (QEH)

(QEH.i) By remaining open to unforeseeable change, a QEH model recognizes ambiguity about which is the correct quantitative model of the process driving outcomes. The defining feature of such change is that it cannot "by any method be [represented *ex ante*] with an objective, quantitatively determined probability" (Knight, 1921, p. 321).

(QEH.ii) Building on Muth's insight, a QEH model represents the market's forecasts of outcomes by assuming that they lie within the intervals within which future outcomes are expected to lie, according to the qualitative expectation implied by the model.

4.1 The Market's Forecast of Dividends

The forecasts of future dividends and prices, $\mathcal{F}_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1})$ in (2) stand for the aggregate values of market participants' forecasts. In order to make these forecasts, participants rely on formal (statistical) methods as well as more informal considerations – namely, their assessments of market sentiment and their own intuitive guesses about the future course of outcomes.

We illustrate how QEH can be used to represent the market's forecast in the context of a model for the dividend process in Section 2. This model formalizes ambiguity by assuming that, although dividends do not unfold according to a stochastic process, they do lie within intervals that vary over time. Moreover, we have used the conditional qualitative expectation to derive the expected intervals for future dividends. QEH represents the market's forecasts of dividends by assuming that they lie within these intervals:

Assumption 1 *Let $\mathcal{F}_t(d_{t+1})$ denote the time-t value of the market's forecast of dividends at $t+1$. QEH represents this forecast as being consistent with the model by assuming that there exists a sequence of coefficients, $\widetilde{b}_t$ such that*

$$\mathcal{F}_t(d_{t+1}) = \widetilde{b}_t x_t \in QE_t\left(\mathcal{I}^d_{t+1}\right) \tag{13}$$

where $\widetilde{b}_t \in b_t[L, U]$.

Our model in Section 2 recognizes that an economist faces ambiguity about the value of the impact of future earnings on dividends, b_{t+1}, by stopping short of specifying a mechanism determining specific values that b_{t+1} take within the intervals $b_t \mathcal{I}_{t+1}/x_{t+1}$. Based on this model, QEH formalizes ambiguity market participants face by not specifying how the values of $\widetilde{b}_t$ are determined within the interval $b_t[L, U]$. However, because our model for the dividend process in Section 2 hypothesizes that the impact of earnings on dividends is positive – that is, b_t and L are both positive – model consistency of the market's forecasts implies that

$$\tilde{b}_t > 0.$$

Note that if the model completely assumed away unforeseeable change in the dividend process – either by constraining b_t to be constant, or by specifying a probabilistic structure for its change over time – $\mathcal{F}_t(d_{t+1})$ would equal the conditional expectation of d_{t+1}, i.e. $\mathcal{F}_t(d_{t+1}) = E(d_{t+1}|\mathbf{x}_t) = E(b_{t+1}x_{t+1}|\mathbf{x}_t)$. Given the realization of $\mathbf{x}_t$, an REH counterpart of the model in Section 2 would determine the precise value of the time-t market's forecasts of d_{t+1} .

Thus, REH and QEH can be seen to represent the market's forecast according to very different assumptions concerning how economists and market participants understand change and the uncertainty that it engenders. REH assumes that, in forming their forecasts, market participants ignore the possibility that the process underpinning outcomes might change at times and in ways that no one can fully foresee. Consequently, REH supposes that an economist and market participants face only what Hansen (2013) referred to as "risk conditioned on the model." In contrast, QEH models assume that, in forming their forecasts, an economist and market participants recognize that outcomes might change in unforeseeable ways. Thus, they face both risk and Knightian uncertainty, which arises from ambiguity about which is the correct model of future outcomes.

4.2 The Market's Forecast of Prices

We formalize ambiguity about which is the correct quantitative model of the process driving prices with the following assumption:

Assumption 2 *The price p_t that the market sets according to the no-arbitrage in (2) lies within an interval denoted by $\mathcal{I}_t^p$, that is,*

$$p_t \in \mathcal{I}_t^p \quad \textit{for } t = 1, 2, 3, ...$$

Next, recall that $QE_t\left(\mathcal{I}_{t+1}^p\right)$ defines the expected interval for p_{t+1} according to the model. Consequently, QEH (QEH.ii) represents the market's time-t forecast of p_{t+1} to lie within this interval, that is:

$$\mathcal{F}_t(p_{t+1}) \in QE_t\left(\mathcal{I}_{t+1}^p\right) \tag{14}$$

5 The No-Arbitrage Condition Under Model Ambiguity

Testable implications of the no-arbitrage condition in (2) are typically derived in the context of REH models. These models represent $\mathcal{F}_t\left(d_{t+1}\right)$ and $\mathcal{F}_t\left(p_{t+1}\right)$, with a conditional expectation of the single probability distribution that, according to the model, represents how dividends and prices unfold over time. Thus, given the information at time t, $\mathbf{x}_t$, the model determines the precise values of the market's forecasts, $\mathcal{F}_t\left(d_{t+1}\right) = E\left(d_{t+1}|\mathbf{x}_t\right)$ and $\mathcal{F}_t\left(p_{t+1}\right) = E\left(p_{t+1}|\mathbf{x}_t\right)$. With these representations of the market's forecasts, the no-arbitrage condition in (2) determines the precise value of the price at each point in time:

$$p_t = \gamma\left[E\left(d_{t+1}|\mathbf{x}_t\right) + E\left(p_{t+1}|\mathbf{x}_t\right)\right] \quad \text{for } t = 1, 2, 3, ... \tag{15}$$

Recognizing ambiguity in a macroeconomic or finance model involving a standard no-arbitrage condition changes the model's implications substantially. Because the model does not determine the precise values of the market's forecasts, $\mathcal{F}_t\left(d_{t+1}\right)$ and $\mathcal{F}_t\left(p_{t+1}\right)$, it does not determine the precise value of the price the market sets. Instead, QEH represents the price p_t set according to the no-arbitrage condition in (2) as lying within the following interval,

$$p_t = \gamma\left[\mathcal{F}_t\left(d_{t+1}\right) + \mathcal{F}_t\left(p_{t+1}\right)\right] \in \gamma\left[QE_t\left(\mathcal{I}_{t+1}^d\right) + QE_t\left(\mathcal{I}_{t+1}^p\right)\right]$$

where we used the QEH representations in (13) and (14): $\mathcal{F}_t\left(d_{t+1}\right) \in QE_t\left(\mathcal{I}_{t+1}^d\right)$ and $\mathcal{F}_t\left(p_{t+1}\right) \in QE_t\left(\mathcal{I}_{t+1}^p\right)$.

Under REH, conditional on $\mathbf{x}_t$, the representation of the no-arbitrage condition in (15) determines the precise relationship between the price and earnings at each point in time. This is because an REH model, by representing the dividend and price processes with a single probability distribution, assumes away ambiguity.

In contrast, under ambiguity, the model no longer represents how dividends and prices unfold over time with a single probability distribution. However, as we show next, a QEH model can relate the market price to earnings in a way that is consistent with QEH's formalization of ambiguity. This representation rests on the following QEH-based interval condition – the counterpart to the no-arbitrage condition in (2):

Assumption 3 *An interval $\mathcal{I}_t^p$ is said to satisfy a no-arbitrage interval condition if*

$$p_t \in \mathcal{I}_t^p \text{ and } \mathcal{I}_t^p \subseteq \gamma\left[QE_t\left(\mathcal{I}_{t+1}^d\right) + QE_t\left(\mathcal{I}_{t+1}^p\right)\right] \tag{16}$$

We refer to $\mathcal{I}_t^p$ as a no-arbitrage interval.

6 The Stock Price and Earnings at a Point in Time

We now use the no-arbitrage interval condition in (16) to derive the relationship between prices and corporate earnings under ambiguity. Applying (16) at $t+1$ we have

$$\mathcal{I}_{t+1}^{p} \subseteq \gamma QE_{t+1}\left(\mathcal{I}_{t+2}^{d}\right) + \gamma QE_{t+1}(\mathcal{I}_{t+2}^{p})$$

Thus,

$$\begin{aligned}\mathcal{I}_{t}^{p} &\subseteq \gamma QE_{t}\left(\mathcal{I}_{t+1}^{d}\right) + \gamma QE_{t}\left(\mathcal{I}_{t+1}^{p}\right)\\ &\subseteq \gamma QE_{t}\left(\mathcal{I}_{t+1}^{d}\right) + \gamma^{2} QE_{t}QE_{t+1}(\mathcal{I}_{t+2}^{d}) + \gamma^{2} QE_{t}QE_{t+1}\left(\mathcal{I}_{t+2}^{p}\right)\\ &\subseteq \gamma QE_{t}\left(\mathcal{I}_{t+1}^{d}\right) + \gamma^{2} QE_{t}^{(1)}\left(\mathcal{I}_{t+2}^{d}\right) + \gamma^{2} QE_{t}^{(1)}\left(\mathcal{I}_{t+2}^{p}\right).\end{aligned}$$

Iterating n times we find that

$$\mathcal{I}_{t}^{p} \subseteq \sum_{k=1}^{n} \gamma^{k} QE_{t}^{(k-1)}\left(\mathcal{I}_{t+k}^{d}\right) + \gamma^{n} QE_{t}^{(n-1)}\left(\mathcal{I}_{t+n}^{p}\right). \tag{17}$$

Finally, in order to derive the representation for the market price, we need a transversality assumption on $\gamma^{n} QE_{t}^{(n-1)}(\mathcal{I}_{t+n}^{p})$ and γ:

Assumption 4 *Interval Transversality condition: Assume that*

$$\gamma U < 1, \tag{18}$$

where U is defined in (11) and that

$$\gamma^{n} QE_{t}^{(n-1)}(\mathcal{I}_{t+n}^{p}) \to 0 \text{ as } n \to \infty. \tag{19}$$

Remark 1 *Obviously the condition $\gamma U < 1$ depends on the distribution of the process x_t. For x_t given by (4) with Gaussian innovations ε_{xt}, the quantities L and U are calculated in Lemma 6 in Appendix B, where it can be seen that $U \simeq 1 + 0.8\sigma_x$ for small values of σ_x^2.*

We summarize the above considerations in the following theorem:

Theorem 1 *Under Assumption 4 and QEH, the no-arbitrage interval $\mathcal{I}_t^p$ in (16), satisfies the following interval relationship,*

$$\mathcal{I}_{t}^{p} \subseteq \sum_{k=1}^{\infty} \gamma^{k} QE_{t}^{(k-1)}\left(\mathcal{I}_{t+k}^{d}\right) = \sum_{k=1}^{\infty} \gamma^{k} QE_{t}\left(\mathcal{I}_{t+k}^{d}\right) = b_t x_t [L_\gamma, U_\gamma], \quad t = 1, 2, 3... \tag{20}$$

where $L_\gamma = \gamma L/(1-\gamma L)$ *and* $U_\gamma = \gamma U/(1-\gamma U)$ *and we used* $QE_t^{(k-1)}\left(\mathcal{I}_{t+k}^d\right) = QE_t\left(\mathcal{I}_{t+k}^d\right)$.

The proof is given in the Appendix.

An immediate corollary to Theorem 1 represents the price p_t as follows:

Corollary 1 *Under the assumptions of Theorem 1, there exists a coefficient* $\theta_t \in b_t[L_\gamma, U_\gamma]$ *such that for* $p_t \in \mathcal{I}_t^p$ *it holds that*

$$p_t = \theta_t x_t \in \mathcal{I}_t^p, \quad t = 1, 2, 3... \tag{21}$$

Theorem 1 and Corollary 1 show that, despite allowing for ambiguity in the dividend and price processes, the model relates the stock price p_t to corporate earnings x_t in a way that is consistent with the representation of the dividend process and the no-arbitrage condition in (2). Moreover, because the coefficient θ_t is positive and x_t is positive, the model characterizes the stock price, p_t, as always positive.

6.1 Representing the Price in Terms of the Market's Forecasts

In an REH model, model-consistency implies a specific value for the coefficient θ_t, and thereby a specific value for p_t. In contrast, consistency in a QEH model implies that θ_t lies within the interval $b_t[L_\gamma, U_\gamma]$, but it does not determine the specific value θ_t takes. Importantly, this allows a QEH model to impose additional restrictions on changes in θ_t over time, while maintaining a model-consistent representation of the market's forecasts. But, in order to do so, the model must relate θ_t to the market's forecasts of dividends and prices.

From (21), the price p_{t+1} set by the market according to the no arbitrage condition in (2) at $t+1$ can be represented as

$$p_{t+1} = \theta_{t+1} x_{t+1} \in \mathcal{I}_{t+1}^p \tag{22}$$

for some coefficient $\theta_{t+1} \in b_{t+1}[L_\gamma, U_\gamma]$.

The model-consistent forecast of p_{t+1} lies within an interval $QE_t\left(\mathcal{I}_{t+1}^p\right)$, given by:

$$\begin{aligned} QE_t\left(\mathcal{I}_{t+1}^p\right) &\subseteq QE_t\left(b_{t+1} x_{t+1}\left[L_\gamma, U_\gamma\right]\right) \\ &= QE_t\left(b_{t+1} x_{t+1}\right)\left[L_\gamma, U_\gamma\right] \\ &\subseteq b_t\left[L \cdot L_\gamma, U \cdot U_\gamma\right] x_t \end{aligned}$$

Under QEH, we represent $\mathcal{F}_t\left(p_{t+1}\right)$ to lie within the $QE_t\left(\mathcal{I}_{t+1}^p\right)$ interval, which we formalize by the following assumption:

Assumption 5 *QEH represents the market's forecast of* p_{t+1} *as being consistent with the model by assuming that there exists a sequence of coefficients,* $\widetilde{\theta}_t$*, such that*

$$\mathcal{F}_t\left(p_{t+1}\right) = \widetilde{\theta}_t x_t \in QE_t\left(\mathcal{I}^p_{t+1}\right), \tag{23}$$

where $\widetilde{\theta}_t \in b_t\left[L \cdot L_\gamma, U \cdot U_\gamma\right]$.

Substituting the representations in Assumption 1 and Assumption 5 into the no-arbitrage condition in (2) implies the following representation of p_t in terms of the time-t representations of the market's forecasts of dividends and prices at $t+1$:

$$p_t = \gamma(\widetilde{b}_t + \widetilde{\theta}_t)x_t$$

We can summarize the foregoing argument in the Corollary to Theorem 1:

Corollary 2 *Under the assumptions of Theorem 1, Assumption 1, and 5, the QEH implied price* p_t *that satisfies the no-arbitrage condition in (2) at all* t *is given by*

$$p_t = \gamma\left[\mathcal{F}_t\left(d_{t+1}\right) + \mathcal{F}_t\left(p_{t+1}\right)\right] = \gamma(\widetilde{b}_t + \widetilde{\theta}_t)x_t = \theta_t x_t, \quad t = 1, 2, 3... \tag{24}$$

where $\widetilde{\theta}_t$ *and* $\widetilde{b}_t$ *are defined in (23) and (13).*

Note that (24) implies that θ_t can be represented in terms of $\tilde{\theta}_t$ and $\tilde{b}_t$:

$$\theta_t = \gamma(\widetilde{b}_t + \widetilde{\theta}_t). \tag{25}$$

As we show in Section 8, this representation enables us to formalize the qualitative effect of non-fundamental factors, such as market sentiment, on the market's forecasts of dividends and prices. Because it recognizes ambiguity, the QEH model can account for the role of both fundamental factors and psychological considerations in model-consistent representations of forecasting, and thereby their role in price movements.

7 Co-movement of Earnings, Dividends, and Prices

Over time, the market revises its forecasts of dividends and prices by taking into account new information on corporate earnings as well as by altering the weights $\tilde{b}_t$ and $\widetilde{\theta}_t$ that it attaches to these earnings. These forecast revisions drive movements in the stock price, p_t. while the model-consistent interval for the stock price, $\mathcal{I}^p_t$, is driven by changes in earnings and the impact coefficients b_t. However, as we show with an example, model-consistency does not guarantee positive co-movement between p_t and x_t. This is in contrast to the positive co-movement between $E_t(d_t|\mathbf{x}_t)$ and x_t implied by the moderate-change condition imposed on changes in b_t.

To examine the co-movement of prices and earnings, we first give a few results for positive co-movement of general sequences. We then derive in Theorem 3 two conditions for the coefficients θ_t, which imply a positive co-movement between p_t and x_t.

7.1 Positive Co-movement

Positive co-movement between sequences is defined as follows:

Definition 2 *For sequences $x_t \geq 0$ and $y_t \geq 0$, we say that they co-move positively from time t to $t+1$, if*

$$\Delta x_{t+1} \Delta y_{t+1} \geq 0. \tag{26}$$

An immediate consequence of the definition is that if x_t increases then y_t increases, and if x_t decreases then y_t decreases. Moreover, provided that $\Delta x_{t+1} \neq 0$, we have $\Delta y_{t+1}/\Delta x_{t+1} \geq 0$ as in (1).

We collect some results on positive co-movement in the next lemma. The proof is given in Appendix A.

Lemma 3 *If the sequences $z_t \geq 0$ and $y_t \geq 0$ co-move positively with $x_t \geq 0$, then:*

(i)	*Positive homogeneity*	*For $\lambda \geq 0$, λz_t co-moves positively with x_t,*
(ii)	*Additivity*	*$z_t + y_t$ co-moves positively with x_t,*
(iii)	*Multiplicativity*	*$z_t y_t$ co-moves positively with x_t.*

Now we can state the result that model consistency in the QEH model is insufficient to imply positive co-movement between prices and earnings at all points in time. We recall from by Lemma 1 that the MC condition on b_t with respect to x_t implies positive co-movement between expected dividends and earnings. However, as the following theorem shows, the MC condition in (8) is not sufficient to ensure positive co-movement between prices and earnings.

The MC condition in (8) is not sufficient to ensure positive co-movement between prices p_t and earnings x_t in the model-consistent representation $p_t = \theta_t x_t$ in (21).

Theorem 2 *If the change Δx_{t+1} satisfies*

$$0 < \tfrac{\Delta x_{t+1}}{x_t} < \tfrac{U-L}{2L},$$

then there exists a $p_t \in b_t x_t[L_\gamma, U_\gamma]$ *and* $p_{t+1} \in b_{t+1}x_{t+1}[L_\gamma, U_\gamma]$ *such that* $\Delta p_{t+1} < 0$. *That is,* p_t *and* x_t *do not co-move positively from time* t *to* $t+1$.

The proof of Theorem 2 is given in Appendix A.

Although model-consistency does not imply positive co-movement, prices and earnings will co-move positively during periods in which the change in θ_t is such that the effect of $\Delta\theta_t$ on Δp_t does not outweigh the effect of Δx_t. To see this, note that the change of the stock price p_t can be written in terms of changes in θ_t and x_t, $\Delta p_{t+1} = \Delta\theta_{t+1}x_{t+1} + \theta_t \Delta x_{t+1}$. In the theorem below, we state two conditions for θ_t which, together, are sufficient for positive co-movement between p_t and x_t.

Theorem 3 *Under the QEH assumptions, and with the price given by (21)*

$$p_t = \theta_t x_t,\ \theta_t \in b_t[L_\gamma, U_\gamma]$$

we find

$$\begin{aligned}
&(i)\ \textit{if}\ \Delta x_{t+1} \geq 0\ \textit{and}\ \Delta(\theta_{t+1}/b_{t+1}) \geq 0,\ \textit{then}\ \Delta p_{t+1} \geq 0,\\
&(ii)\ \textit{if}\ \Delta x_{t+1} \leq 0\ \textit{and}\ \Delta(\theta_{t+1}/b_{t+1}) \leq 0,\ \textit{then}\ \Delta p_{t+1} \leq 0.
\end{aligned}$$

The proof of Theorem 3 is given in the Appendix A, but note here that Theorem 1 implies that $L_\gamma \leq \theta_t/b_t \leq U_\gamma$, and similarly at time $t+1$, that is, such that

$$\theta_{t+1}/b_{t+1} \in [L_\gamma, \theta_t/b_t] \cup [\theta_t/b_t, U_\gamma]\,.$$

Thus, if θ_{t+1}/b_{t+1} lies within the interval $[L_\gamma, \theta_t/b_t]$, then $\Delta(\theta_{t+1}/b_{t+1}) \leq 0$. Similarly, if θ_{t+1}/b_{t+1} lies within the interval $[\theta_t/b_t, U_\gamma]$, then $\Delta(\theta_{t+1}/b_{t+1}) \geq 0$, which motivates the conditions (i) and (ii) in Theorem 3.

The theorem states that during periods of time when the changes in earnings x_t are positive (negative) and the changes in the ratio θ_t/b_t are positive (negative), the changes in prices p_t are positive (negative). Stated differently, if $\Delta x_{t+1}\Delta(\theta_{t+1}/b_{t+1}) \geq 0$, then we find from (i) that $\Delta x_{t+1} \geq 0$ implies $\Delta p_{t+1} \geq 0$, and from (ii) that $\Delta x_{t+1} \leq 0$ implies $\Delta p_{t+1} \leq 0$, so that $\Delta x_{t+1}\Delta p_{t+1} \geq 0$. We formulate this as a corollary:

Corollary 3 *In the QEH model, during periods in which* θ_t/b_t *and* x_t *co-move positively from time* t *to* $t+1$, *it follows that* p_t *and* x_t *co-move positively.*

Positive co-movement between prices and earnings follows from the assumption that θ_t/b_t and x_t co-move positively. Analogous to the moderate change condition for b_t in (6) and

(8), the following assumption states the stronger condition that θ_t changes moderately with respect to x_t:

Assumption 6 *In addition to lying within the model-consistent interval, $\theta_t \in b_t[L_\gamma, U_\gamma]$, we assume that θ_t changes moderately with respect to x_t,*

$$\frac{|\Delta\theta_{t+1}|}{\theta_t} \leq \frac{|\Delta x_{t+1}|}{x_{t+1}} \tag{27}$$

such that

$$\theta_{t+1} \in b_{t+1}\left[L_\gamma, U_\gamma\right] \cap \theta_t \mathcal{I}_{t+1}/x_{t+1}. \tag{28}$$

It follows from Assumption 6 that,

$$p_{t+1} = \theta_{t+1} x_{t+1} \in \theta_t \mathcal{I}_{t+1}, \text{ where } \mathcal{I}_{t+1} = \left[(x_{t+1} - |\Delta x_{t+1}|)^+, (x_{t+1} + |\Delta x_{t+1}|)\right]. \tag{29}$$

The Assumption 6 restricts θ_{t+1} to an interval around θ_t, so it constrains the interval for p_{t+1}, given its past observation p_t, to a sub-interval of the model-consistent interval. Thus, 6 restricts the change in the stock price in a way that implies positive co-movement with earnings.

Corollary 4 *Under Assumption 6, $p_t = \theta_t x_t$ and x_t co-move positively.*

8 Fundamentals and Market Sentiment in Stock-Price Movements

Relying on model consistency, we have related stock prices to fundamental factors, specifically earnings, and we have represented these prices in terms of the market's forecast of dividends and prices next period in Corollary 2. However, as we have shown, model consistency alone does not ensure positive co-movement between prices and earnings. This opens the possibility that the model can accord a role to both fundamental and non-fundamental factors in driving prices, while representing the market's forecasts as being consistent with the model's representations of the processes underpinning outcomes.

We illustrate this possibility by modifying Assumption 6, such that the change in θ_t satisfies the moderate-change condition with respect to x_t, but the direction of the change in θ_t depends on market sentiment, denoted by some index i_t, and the changes in earnings. Specifically, we assume that when the market is optimistic, which we formalize with $i_t > 0$, and $\Delta x_t > 0$, the market maintains or revises upward its expectation of the future impact of earnings on prices, $\widetilde{\theta}_t$ and/or dividends $\widetilde{b}_t$. As a result θ_t either remains unchanged or increases. Similarly, when the market is pessimistic and earnings decrease, it maintains or revises downward its expectation of the future impact of earnings on prices, $\widetilde{\theta}_t$ and/or on dividends, $\widetilde{b}_t$. As a result θ_t either remains unchanged or decreases. We formalize this with the following assumption:

Assumption 7 *Let i_t denote a sentiment index, such that $i_t > 0$ indicates market optimism, and $i_t < 0$ pessimism. Let θ_t in (21) satisfy the MC Assumption 6, but modified by the market sentiment in the following way:*

$$\begin{array}{ll} (i) \text{ if } i_{t+1} > 0 \text{ and } \Delta x_{t+1} > 0, \text{ then} & \theta_{t+1} \in b_{t+1}\left[L_\gamma, U_\gamma\right] \cap \theta_t \mathcal{I}_{t+1}^{+}/x_{t+1}, \\ (ii) \text{ if } i_{t+1} < 0 \text{ and } \Delta x_{t+1} < 0, \text{ then} & \theta_{t+1} \in b_{t+1}\left[L_\gamma, U_\gamma\right] \cap \theta_t \mathcal{I}_{t+1}^{-}/x_{t+1}, \\ (iii) \text{ otherwise,} & \theta_{t+1} \in b_{t+1}\left[L_\gamma, U_\gamma\right] \cap \theta_t \mathcal{I}_{t+1}/x_{t+1}, \end{array}$$

where $\mathcal{I}_{t+1}^{+} = \left[x_{t+1}, x_{t+1} + |\Delta x_{t+1}|\right]$ and $\mathcal{I}_{t+1}^{-} = \left[\left(x_{t+1} - |\Delta x_{t+1}|\right)^{+}, x_{t+1}\right]$.

If follows from Assumption 7 that if $i_{t+1} > 0$ and $\Delta x_{t+1} \geq 0$, then

$$p_{t+1} = \theta_{t+1} x_{t+1} \in \theta_t \mathcal{I}_{t+1}^{+}, \tag{30}$$

and if $i_{t+1} < 0$ and $\Delta x_{t+1} \leq 0$, then

$$p_{t+1} = \theta_{t+1} x_{t+1} \in \theta_t \mathcal{I}_{t+1}^{-}. \tag{31}$$

Under (i), (ii) or (iii) in Assumption 7, the stock price co-moves positively with earnings, as θ_t always satisfies the moderate-change condition in (27). But during periods when the market is optimistic and earnings increase, θ_t is assumed to increase. Thus, the market's optimism is assumed to lead it to reinforce the positive effect of earnings on the stock price, which is formalized with an increase in θ_{t+1}. Similarly, when the market is pessimistic and earnings decrease, the assumed fall in θ_{t+1} reinforces the negative effect of earnings on the stock price. We state this as a Corollary:

Corollary 5 *Under Assumption 7 the stock price p_t co-moves positively with earnings x_t. Moreover, during the periods in which either (i) or (ii) in Assumption 7 hold, the sentiment i_t amplifies the effect of change in earnings on the stock price.*

9 An Econometric Investigation of the QEH Model

The QEH model assumes that the parameters b_t, $\tilde{b}$, and $\tilde{\theta}_t$ (and thus also θ_t) lie within stochastic intervals, such that dividends and stock prices lie within stochastic intervals as well. Consequently, the QEH model does not specify a complete stochastic process for d_t and x_t (and thus for p_t), which can be directly estimated and tested empirically.

In order to assess the empirical adequacy of the QEH model's assumptions or its qualitative implications, we propose a statistical model for d_t and x_t. The model captures the dynamics of x_t and d_t, and represents b_t with a time-varying random coefficient β_t. Specifically, building on the Generalized Autoregressive Score (GAS) approach, we consider

$$d_t = \beta_t x_t + u_t \tag{32}$$

$$\Delta \log x_t = -\sigma_x^2/2 + \varepsilon_{xt}, \tag{33}$$

where

$$\beta_t = \varphi\left(\beta_{t-1}, x_{t-1}; \tau\right), \tag{34}$$

where u_t is an i.i.d.$(0, \sigma_u^2)$ sequence with density $p(\cdot)$, ε_{xt} is an i.i.d.$(0, \sigma_x^2)$ sequence, $\varphi(\cdot)$ is a link function, and τ is a vector of parameters. Blasques *et al* (2014a) and Blasques *et al* (2015) show that an optimal choice of the link function $\varphi(\cdot)$ is given by

$$\beta_t = \omega + \phi\beta_{t-1} + \alpha s_{t-1}, \tag{35}$$

where s_t is the (possibly scaled) score of the model,

$$s_t = \delta\left(\beta_t\right)^{-1} \partial \log \ell\left(d_t | x_t, \beta_t; \theta\right) / \partial\beta_t,$$

and $\delta(\beta_t)$ is the scaling factor. Assuming that u_t is Gaussian, or that it is $t_v(0,1)$ distributed, implies that

$$s_t = \delta\left(\beta_t\right)^{-1}\left(d_t - \beta_t x_t\right) x_t/\sigma^2 \quad \text{and} \quad s_t = \delta\left(\beta_t\right)^{-1} \frac{\left(d_t - \beta_t x_t\right) x_t}{v + \left(d_t - \beta_t x_t\right)^2}, \tag{36}$$

respectively, for some scaling factor $\delta(\beta_t)$. With the scaling factor $\delta(\beta_t)$ chosen as the conditional expectation of the Hessian, that is,

$$E_t\left(\partial^2 \log \ell\left(d_t | x_t, \beta_t; \theta\right) / \partial\beta_t \partial\beta_t\right) = x_t^2/\sigma^2,$$

the Gaussian case implies that $s_{t-1} = \frac{d_{t-1}}{x_{t-1}} - \beta_{t-1}$. For this choice of the scaling factor, the specification for β_t in (35) can be rewritten as

$$\beta_t = \omega + \varphi\beta_{t-1} + \alpha\frac{d_{t-1}}{x_{t-1}}. \tag{37}$$

The model can be estimated by maximum likelihood, and standard misspecification tests can be used to assess its adequacy in representing the data. If the misspecification tests are not rejected, we consider the estimated model a valid representation of the data for the sample period considered. The estimated statistical model can then be used to assess the adequacy of the QEH model's assumptions and implications.

In the reminder of this Section, we illustrate the steps of the statistical analysis outlined above using simulated data that satisfy the assumptions of the QEH model. We leave a detailed development of the econometric methodology for the QEH models for future research.

9.1 QEH-Consistent Simulations

Because a QEH model formalizes ambiguity, there are myriad sequences of $b_t, \tilde{b}_t$ and $\tilde{\theta}_t$ (and thus also θ_t) that are consistent with its representations of the dividend and price processes. Although the model restricts these parameters to lie within their respective intervals, it does not specify a rule or a mechanism determining the values of the parameters within the intervals. Thus, we cannot directly simulate data from the QEH model.

However, given a specific sequence of b_t and θ_t and a chosen set of values for the fixed parameters of the model, we can simulate time series for dividends, earnings, and stock prices. We could, for example, manually choose sequences for b_t and θ_t within their respective intervals at every point in time, or we could draw a sequence for b_t and θ_t from a stochastic process bounded to lie within those intervals. We emphasize that there are myriad such sequences, as well as myriad methods that could be used to simulate data consistent with the QEH model. As our objective here is to sketch our approach, for ease of illustration, as detailed below, we draw sequences b_t and θ_t from the QEH-implied intervals uniformly, conditionally on the past values of these parameters and simulated $\mathbf{x}_t$.

We denote the simulated sequences by b_t^s and θ_t^s. These, together with the QEH model's representations for earnings, dividends, and prices, define a data-generating process (DGP) that can be used to simulate a time series for $(x_t,\, d_t,\, p_t)$ that is consistent with the assumptions of the QEH model. In line with the QEH model, we simulate d_t and x_t in (3) and (4) as follows

$$d_t = b_t^s x_t + \varepsilon_{dt}, \text{ and } \Delta \log x_t = -\sigma_x^2/2 + \varepsilon_{xt}, \tag{38}$$

with ε_{dt} i.i.d. $N(0, \sigma_d^2)$, ε_{xt} i.i.d. $N(0, \sigma_x^2)$, and the values for σ_x^2 and σ_d^2 specified below.

Next, to simulate a sequence b_t^s satisfying the MC condition in (6), we consider the stochastic specification given by,

$$b_t^s = b_{t-1}^s \varepsilon_{bt}, \text{ with } \varepsilon_{bt} \text{ i.i.d. uniform on } \mathcal{I}_t / x_t. \tag{39}$$

Hence, we draw the sequence of b_t^s uniformly within the interval for b_t defined in (7). This yields one potential QEH-consistent sequence of b_t^s. Given this sequence, we simulate QEH-consistent series for dividends.

As detailed below, we next draw three different sequences of θ_t^s, denoted θ_{it}^s for $i = 1, 2, 3$, which lie within the intervals for θ_t specified in Theorem 1, Assumption 6, and Assumption 7, respectively. Given the sequences of θ_{it}^s, we simulate three different stock prices consistent with the QEH model under the three different assumptions about the interval for θ_t. The simulations allow us to illustrate the effect of adding the MC condition on θ_t in Assumption 6, as well as to illustrate how sentiment can drive the price based on Assumption 7.

Specifically, given the simulated x_t and θ_{it}^s, we simulate the stock prices as

$$p_{it} = \theta_{it}^s x_t, \text{ with } \theta_{it}^s \text{ i.i.d. uniform on } \mathcal{I}_{it}^\theta,$$

for $i = 1, 2, 3$. In the first case, we set

$$\mathcal{I}_{1t}^\theta = b_t^s \left[L_\gamma, U_\gamma \right], \tag{40}$$

so that θ_{1t}^s lies within the model-consistent interval defined in Theorem 1. In the second case, we set

$$\mathcal{I}_{2t}^\theta = b_t^s \left[L_\gamma, U_\gamma \right] \cap \theta_{2t-1}^s \mathcal{I}_t / x_t, \tag{41}$$

such that θ_{2t}^s additionally satisfies the moderate change condition with respect to x_t in Assumption 6. For the final case we introduce a sentiment index, i_t, and define

$$\mathcal{I}_{3t}^\theta = \begin{cases} b_t^s \left[L_\gamma, U_\gamma \right] \cap \theta_{3t-1}^s \mathcal{I}_t^+ / x_t & \text{if } i_t > 0 \text{ and } \Delta x_t > 0, \\ b_t^s \left[L_\gamma, U_\gamma \right] \cap \theta_{3t-1}^s \mathcal{I}_t^- / x_t & \text{if } i_t < 0 \text{ and } \Delta x_t < 0, \\ b_t^s \left[L_\gamma, U_\gamma \right] \cap \theta_{3t-1}^s \mathcal{I}_t / x_t & \text{otherwise,} \end{cases} \tag{42}$$

such that the interval $\mathcal{I}_{3t}^\theta$ depends on market sentiment i_t and the change in earnings as specified in Assumption 7. For simplicity, the sentiment index i_t is here defined as

$$i_t = \mathbf{1}(1 \leq t \leq T_1) - \mathbf{1}(T_1 < t \leq T_2) + \mathbf{1}(T_2 < t \leq T), \tag{43}$$

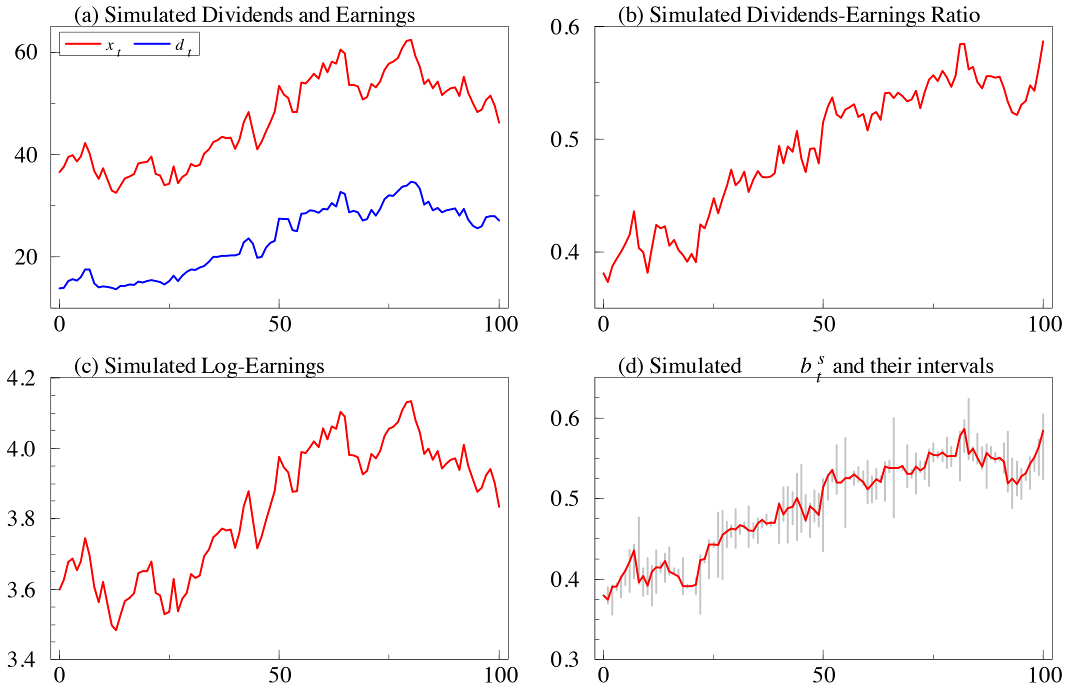

Figure 1: The figure shows the simulated dividends and earnings series from the data-generating process in Definition 3. Panel (a) shows the simulated series x_t and d_t. Panel (b) shows the dividends-earnings ratio, d_t/x_t. Panel (c) shows the simulated series $\log x_t$. Panel (d) shows the simulated series b_t^s (red line), while the vertical grey lines indicate the interval within which b_t^s is uniformly drawn, $b_{t-1}^s \mathcal{I}_t / x_t$, see (8).

where $T = 100, T_1 = T/4, T_2 = 3T/4$ and $\mathbf{1}(\cdot)$ is the indicator function.

We formally define the DGP as follows:

Definition 3 *The DGP of* $\left(d_t, x_t, i_t, (p_{it})_{i=1,2,3}\right)$ *is defined by equations (38)-(43).*

We simulate an effective sample size of $T = 100$ observations from the DGP in Definition 3. We choose parameter values to mimic the Standard & Poor's Composite Stock Price index for the 100 quarterly observations from 1980(1) to 2004(4).[8] Specifically, we set $\sigma_d = 0.2$ and $\sigma_x = 0.05$ corresponding to the estimated standard errors from autoregressive models for the time-series data for dividends and log-earnings, respectively. Moreover, we set the initial values $\log x_0 = 3.6$, $b_0 = 0.38$, and $\theta_{i0} = 6.85$ for $i = 1, 2, 3$ corresponding to the values

[8] The data is available from Robert Shiller's website at http://www.econ.yale.edu/~shiller/data.htm. Real measures of the stock price index, earnings, and dividends are computed using the consumer price index (CPI). Monthly data is available, but as the earnings and dividends series are interpolated from quarterly observations we consider only the quarterly observations corresponding to March, June, September, and December.

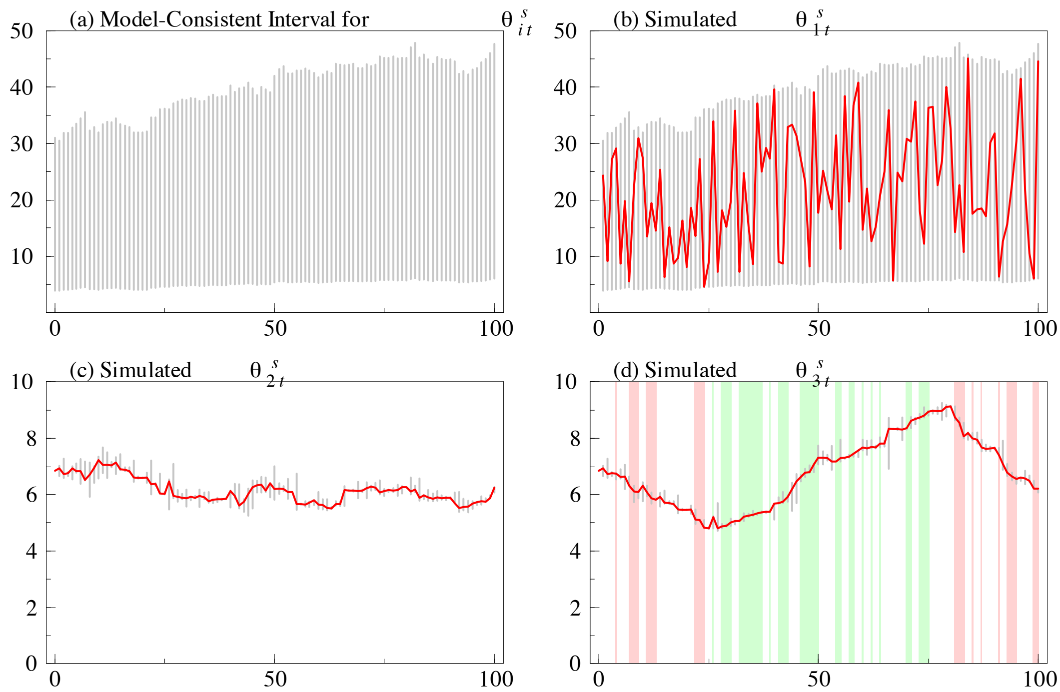

Figure 2: The figure shows the simulated model-consistent interval for θ_{it}^s and the simulated series θ_{it}^s for $i = 1, 2, 3$. Panel (a) shows the model-consistent interval for θ_{it}^s given by $b_t^s x_t[L_\gamma, U_\gamma]$. Panels (b)-(d) shows the simulated series θ_{it}^s for $i = 1, 2, 3$, while the vertical grey lines indicate the intervals $\mathcal{I}_{it}^\theta$, defined in (40)-(42), within which θ_{it}^s is uniformly drawn. The simulated θ_{1t}^s in Panel (b) lies in the model-consistent interval defined in Theorem 1. The simulated θ_{2t}^s in Panel (c) also satisfies the moderate change condition with respect to x_t as defined in Assumption 6. Finally, the simulated θ_{3t}^s in Panel (d) depends on sentiment as defined in Assumption 7. The light green shading indicates periods where $s_t > 0$ and $\Delta x_t > 0$, while the light red shading indicates periods where $s_t < 0$ and $\Delta x_t < 0$. Note that the scales differ between the upper and lower panels.

of log-earnings, the dividend-earnings ratio, and the price-earnings ratio in 1980(1). Finally, we set $\gamma = 0.95$. The chosen parameter values imply that $L = 0.960$, $U = 1.040$, $L_\gamma = 10.37$, and $U_\gamma = 81.61$. For comparability between θ_{it}^s and p_{it}, we use the same sequence of random shocks to simulate θ_{it} for $i = 1, 2, 3$.

Figure 1 shows the simulated dividends and earnings series, as well as the simulated coefficients b_t^s. Panel (a) in Figure 2 shows the model-consistent intervals for θ_{it}^s given by $b_t^s[L_\gamma, U_\gamma]$, while Panel (a) in Figure 3 shows the model-consistent interval for p_{it} given by $b_t^s x_t[L_\gamma, U_\gamma]$. Over time, changes in these intervals are driven by changes in x_t and b_t^s. We note that the upper limit for θ_{it}^s and p_{it} is very large, which is a consequence of the QEH model not imposing an upper limit on b_t.

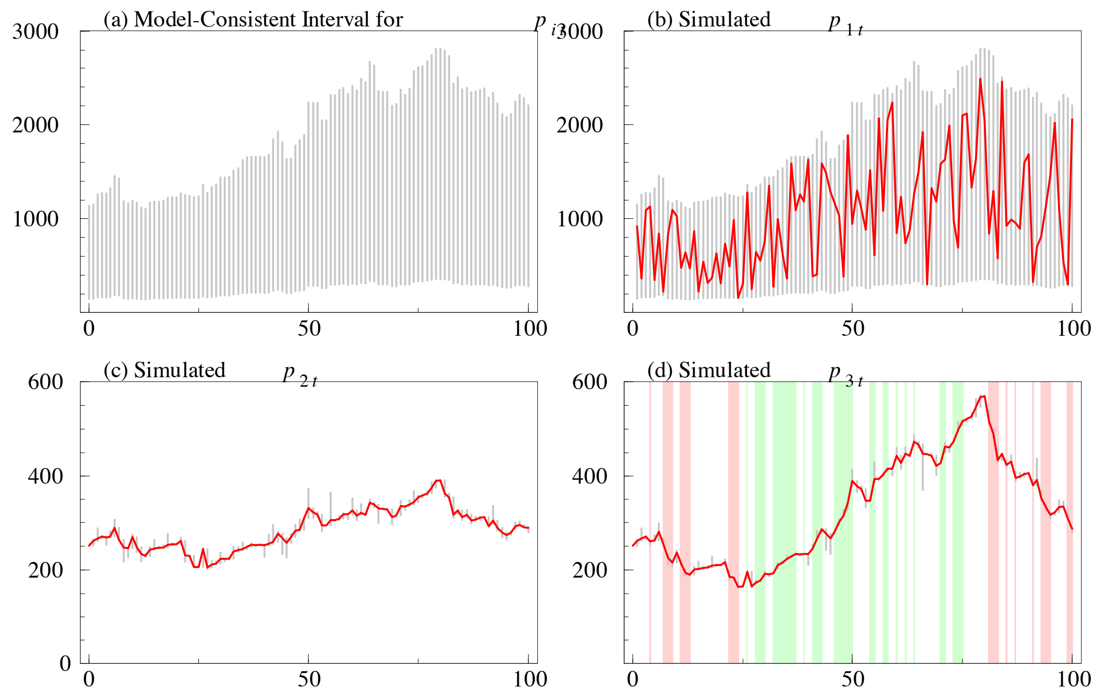

Figure 3: The figure shows the simulated model-consistent interval for the stock price and the simulated stock prices $p_{it} = \theta^s_{it} x_t$ for $i = 1, 2, 3$ based on the data-generating process in Definition 3. Panel (a) shows the simulated model-consistent interval for the stock price, $b^s_t x_t[L_\gamma, U_\gamma]$, as defined in Assumption 1. Panels (b)-(d) shows the simulated series p_{it} for $i = 1, 2, 3$, while the vertical grey lines indicate the intervals within which p_{it} is uniformly distributed. The simulated price p_{1t} in Panel (b) lies in the model-consistent interval defined in Theorem 1. The simulated price p_{2t} in Panel (c) also satisfies the moderate change condition with respect to x_t as defined in Assumption 6. Finally, the simulated price p_{3t} in Panel (d) depends on sentiment as defined in Assumption 7. The light green shading indicates periods where $s_t > 0$ and $\Delta x_t > 0$, while the light red shading indicates periods where $s_t < 0$ and $\Delta x_t < 0$. Note that the scales differ between the upper and lower panels.

Panels (b)-(d) in Figure 2 show the simulated coefficients θ^s_{it} for $i = 1, 2, 3$, while Panels (b)-(d) in Figure 3 show the corresponding simulated prices p_{it}. In the first case, θ^s_{1t} is drawn uniformly from the the model-consistent interval, so p_t is uniformly distributed over the interval $\mathcal{I}^p_t$. In the second case, θ^s_{2t} also satisfies the MC condition with respect to x_t defined in Assumption 6. That implies that changes in θ_{2t} and p_{2t} are much smaller compared to the first case.

The final case, Panel (d) in Figure 3, illustrates an important implication of the model: prices tend to undergo wider swings during the periods in which market optimism (pessimism) coincides with increases (decreases) in earnings. The effect of sentiment is added, so that the realizations of θ^s_{3t} satisfy the restrictions in Assumption 7. During the first T_1

observations, the market is pessimistic ($i_t < 0$), and the coefficient θ^s_{3t} tends to decrease. The coefficient tends to increase for the majority of the period $(T_1, T_2]$ during which the market is optimistic ($i_t > 0$), again tending downward during the last part of the simulated sample during which θ^s_{3t} tends to decrease as $i_t < 0$. Given the choice of i_t, the simulated θ^s_{3t} and p_{3t} exhibit swings that are not present in θ^s_{2t} and p_{2t}. As we discuss in the Introduction, this illustrates the model's potential to explain price swings as being driven primarily by fundamental factors, with psychological considerations playing an amplifying role.

9.2 An Estimation and Testing of the GAS Model for the Simulated Data

For the simulated data, we estimate the GAS model in (32) with the specification of β_t in (37). The model is estimated by maximum likelihood for the sample $t = 1, 2, ..., T$ conditional on the initial values $(d_0, x_0, \beta_0 = d_0/x_0)$. Table 1 shows the estimated coefficients, numerical standard errors, and the standard misspecification tests for no autocorrelation for order four, no autoregressive conditional heteroskedasticity of order four, and normality of the estimated residuals.

	Estimate	*Std. errors*
ω	0.014	0.010
φ	0.170	0.116
α	0.806	0.113
β_0	0.381	
σ_u	0.581	
	Statistics	*p-value*
AR(4)	1.99	0.74
ARCH(4)	2.11	0.72
Normality	2.84	0.24

Table 1: Maximum likelihood estimates of the GAS model in (32) and (37) for the simulated d_t based on the simulation data-generating process in Definition 3. The AR(4) test refers to the test for no autocorrelation of order four by Godfrey (1978). The ARCH(4) test refers to the test for no autoregressive conditional heteroskedasticity of order four by Engle (1982). Normality refers to the Jarque-Bera test. The p-values for the misspecification tests are calculated assuming an asymptotic χ^2 distribution with degrees of freedom 4, 4, and 2, respectively.

The hypotheses of no autocorrelation, no ARCH, and normality are not rejected. This is also evident from Figure 4, which shows misspecification plots for the estimated residuals,

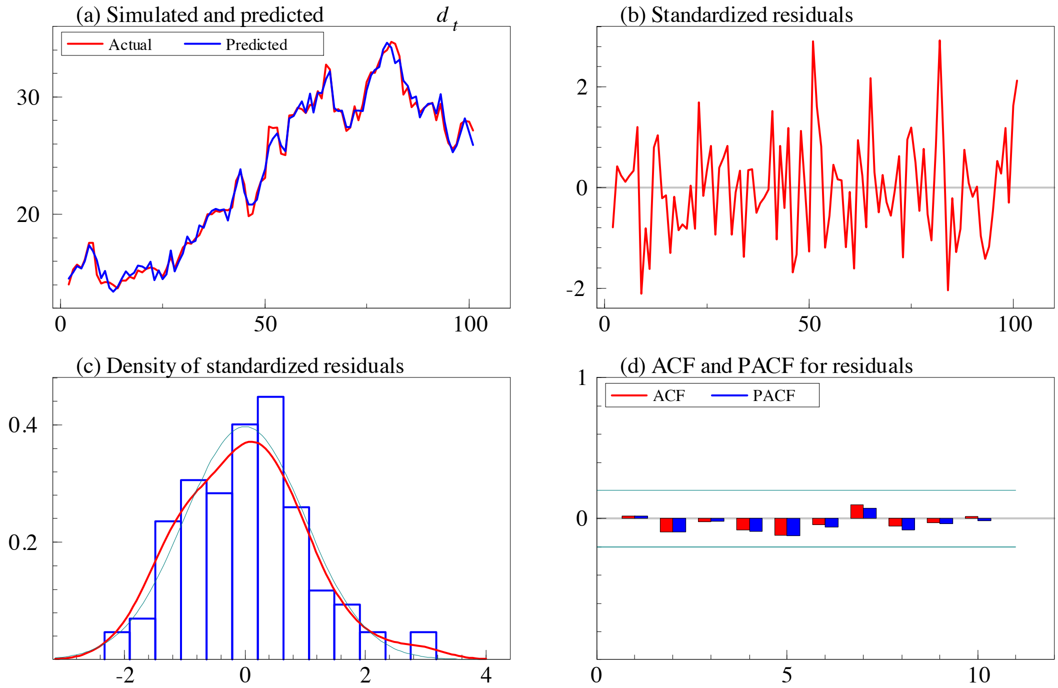

Figure 4: The figure shows the misspecification plots for the estimated residuals, $\widehat{u}_t$, from the model in equations (32) and (37). Panel (a) shows the simulated d_t and the predicted d_t. Panel (b) shows the standardized estimated residuals. Panel (c) shows a histogram of the standardized residuals along with the estimated kernel density function (red line) and the density function from a standard normal distribution for reference. Finally, Panel (d) shows the autocorrelation function (ACF) and partial autocorrelation function (PACF) for the estimated residuals.

$\hat{u}_t = d_t - \hat{\beta}_t x_t$. The estimated residuals appear independent over time and normally distributed. Based on the standard misspecification tests and the plots of the estimated residuals, we conclude that the estimated model is a valid representation of the (simulated) data that is consistent with the QEH model. Crucially, this shows that the random process for β_t in (37) adequately represent the simulated sequence b_t^s, despite the fact that the formulation for β_t is much simpler – involving only three parameters, (ω, ϕ, α) and the initial value β_0 – than the highly non-linear DGP in (39) used to simulate b_t^s.

The estimated sequence $\hat{\beta}_t$ is shown in Figure 5 (upper panel) along with the simulated sequence b_t^s. We note that $\hat{\beta}_t$ follows b_t^s closely, though $\hat{\beta}_t$ leads b_t^s by one period.

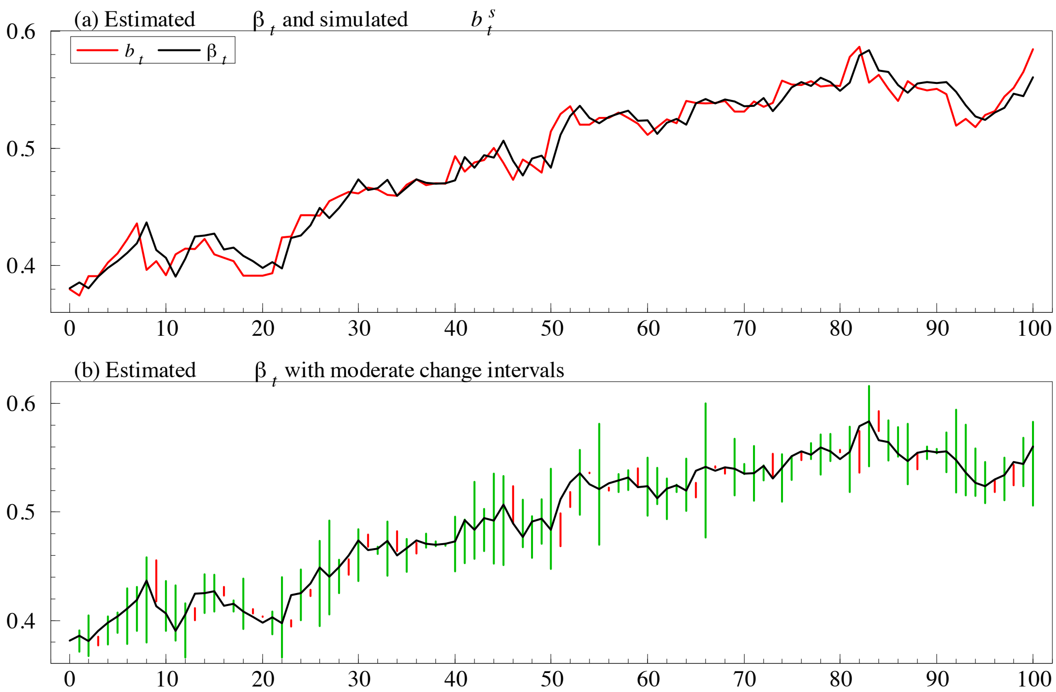

Figure 5: Panel (a) shows the estimated β_t from the model in equations (32) and (37) (black line) and the simulated b_t^s (red line). Panel (b) shows the estimated β_t, while the vertical lines indicate the moderate change intervals for β_t given by $\beta_{t-1}\mathcal{I}_t/x_t$, see (8). The green vertical lines indicate that β_t lies within the interval, while red vertical lines indicate that β_t lies outside the interval.

9.2.1 Assessing the Adequacy of the QEH Model

We now sketch how the estimates of the GAS analog can be used to assess the adequacy of the QEH model's assumptions and implications.

In Table 2, we list some assumptions and implications of the QEH model and how they can be formulated in terms of the coefficients, β_t, and observations, x_t and p_t.

QEH Model	Statistical Model
$b_t > 0$	$\beta_t > 0$
MC of b_t with respect to x_t	$\beta_t \in \beta_{t-1}\mathcal{I}_t/x_t$, $\mathcal{I}_t$ defined in (8)
Positive co-movement between $b_t x_t$ and x_t	$\Delta(\beta_t x_t)\Delta x_t \geq 0$
Positive co-movement between p_t and x_t	$\Delta p_t \Delta x_t \geq 0$
$p_t \in b_t x_t[L_\gamma, U_\gamma]$	$p_t \in \beta_t x_t[L_\gamma, U_\gamma]$

Table 2: The table shows the qualitative assumptions and implications of the QEH model allowing for ambiguity, and their formulations in the statistical model.

The basic assumption of the QEH model is that $b_t > 0$ can be imposed on β_t in the statistical model, or it can left unrestricted. In the latter case, we find $\hat{\beta}_t > 0$ for all observations in the sample. We can check how often the MC condition in (7) is satisfied for $\hat{\beta}_t$ by calculating

$$T^{-1} \sum_{t=1}^{T} \mathbf{1}(\hat{\beta}_t \in \hat{\beta}_{t-1}\mathcal{I}_t/x_t) = 0.71,$$

see also (8). A large value indicates that the development of the unobserved coefficients b_t tends to be moderately changing. On that basis, we conclude that the MC condition tends to hold for $\hat{\beta}_t$.

Panel (b) in Figure 5 provides further support for this conclusion. The black line in the panel shows the estimated β_t and the vertical lines display the MC intervals for $\hat{\beta}$, given by $\hat{\beta}_{t-1}\mathcal{I}_t/x_t$. Green vertical lines in the figure indicate that $\hat{\beta}$ lies within the interval, while red vertical lines indicate that for $\hat{\beta}$ lies outside the interval. The figure shows that whenever $\hat{\beta}_t$ lies outside the MC interval, it is typically not far outside the interval.

It is also possible to incorporate the MC condition for b_t in the statistical model as an MC condition for β_t. This would yield a highly non-linear statistical model. Inference in such a model needs to be developed in detail, for example by extending the bootstrap theory in Cavaliere *et al* (2012, 2015).

We find that $\hat{\beta}_t x_t$ co-moves positively with x_t in 83% of the observations, so we conclude that $\hat{\beta}_t x_t$ and x_t tend to co-move positively.

We simulated three different price series, $p_{it} = \theta^s_{it} x_t$, with θ^s_{it} corresponding to the intervals (40), (41), and (42) respectively. Calculating $T^{-1} \sum_{t=1}^{T} \mathbf{1}(p_{it} \in \hat{\beta}_t x_t[\hat{L}_\gamma, \hat{U}_\gamma])$ yields 99%, 96%, and 100% for the three price series. In all cases, we conclude that the prices p_{it} lie in the estimated model-consistent interval given by $\hat{\beta}_t x_t[\hat{L}_\gamma, \hat{U}_\gamma]$.

We can assess the moderate change of θ_t with respect to x_t in Assumption 6 directly from the observations for p_t and x_t. To do so, we replace θ_t by p_t/x_t in (29), and compute $T^{-1} \sum_{t=1}^{T} \mathbf{1}\left(p_{it} \in (p_{it-1}/x_{t-1})\mathcal{I}_t\right)$. For the three prices, we find 6%, 100%, and 100% respectively. The two latter results reflect that both p_{2t} and p_{3t} have been simulated to satisfy the MC condition. However, the result for p_{1t} indicates that simulating the price from the interval $b^s_t x_t[L_\gamma, U_\gamma]$ without the MC condition yields positive co-movement in only 6% of the observations. So despite the simplicity of the measure, it would correctly lead us to conclude that θ_t is not moderately changing with respect to x_t.

We can assess the amplifying effect of sentiment in Assumption 7 (i) and (ii) by replacing θ_t with p_t/x_t in (30) and (31). For those observations where $i_t > 0$ and $\Delta x_t \geq 0$, we find that $p_{it} \in (p_{it-1}/x_{t-1})\,\mathcal{I}^+_t$ in 0%, 43%, and 100% of the observations for the three prices, respectively. Similarly, we find that $p_{it} \in (p_{it-1}/x_{t-1})\,\mathcal{I}^-_t$ in 0%, 48%, and 100% of the observations where $i_t < 0$ and $\Delta x_t \leq 0$. The results for p_{3t} reflect that this price has been simulated to satisfy Assumption 7. However, for the first two prices, we would correctly reject the amplifying effect of sentiment.

These examples illustrate how the estimates yielded by the GAS model and the observed time series can be used to assess the QEH model's assumptions and implications.

10 Concluding Remarks

REH models assume that neither an economist nor market participants face ambiguity about which is the correct model of outcomes. As a result, they represent the market's forecasts with a conditional expectation of the single probability distribution implied by an economist's model. Hansen (2013, p. 399) argues REH representations thus "miss something essential: uncertainty [arising from] ambiguity about which is the correct model" of the process underpinning outcomes. However, recognizing ambiguity requires jettisoning the assumption that change can be represented with a probabilistic rule and that a single conditional distribution represents how future outcomes actually unfold over time.

This paper has developed a novel mathematical framework that formalizes the ambiguity an economist faces, by opening the model to unforeseeable change. The defining feature of such change is that it cannot "by any method be [represented ex ante] with an objective, quantitatively determined probability" (Knight, 1921, p. 321).

Leaving an economist's model open to unforeseeable change is the key to representing forecasts by market participants in a way that is consistent with the economist's hypothesis that the process driving outcomes can change at times and in ways that cannot be foreseen by anyone with a probabilistic rule. We call our approach to representing forecasts by market participants who face ambiguity about which is the correct quantitative model of this process the Qualitative Expectations Hypothesis (QEH).

By recognizing ambiguity, a QEH model can account for the role of both the fundamental factors, on which REH models focus, and the psychological factors underpinning behavioral-finance models. And, it can do so without abandoning model consistency.

This feature of QEH models contrasts with behavioral-finance models' jettisoning of model consistency. Indeed, once we recognize that an economist and market participants face ambiguity, the common belief that participants who rely on psychological considerations forego profit opportunities appears to be an artifact of behavioral-finance models' probabilistic representations of change. We leave for future research the QEH-based analysis of the role of psychological considerations in rational forecasting.

QEH's ability to represent the role of psychological factors in model-consistent forecasting illustrates how leaving the asset-price models open to ambiguity may shed new light on long-standing puzzles in financial economics. The plot of the simulated data in Panel (d) in Figure 3 provides an example of one of the QEH model's novel implications: stock prices tend to undergo wider swings during periods in which market optimism (pessimism) coincides with increases (decreases) in earnings.

This illustration suggests the QEH model's potential to explain why stock prices "fluctuate too much to be justifie" by REH-based market forecasts of dividends (Shiller, 1981). This puzzle has provided the raison d'être for the behavioral-finance approach, which represents price swings as being driven primarily by psychological factors that are largely unrelated to fundamental factors. By according both earnings and market sentiment a role in model-consistent representations of forecasting, QEH points toward a way to explain price swings as being driven primarily by fundamental factors, with psychological considerations playing an amplifying role.

However, in order to assess the QEH model's ability to explain asset-price swings and other puzzles, we need to develop an econometric methodology for models that are open to unforeseeable change and apply it to actual, rather than simulated, time-series data. Although the sketch of this methodology presented here, together with its application to the simulated data, appears promising, full development of this methodology remains an important topic for future research.

A Proofs of Theorems and Lemmas

Proof of Lemma 1. The changes in expected dividends in (10) can be expanded as follows,

$$\Delta E(d_t|\mathbf{x}_t) = x_{t+1}\Delta b_{t+1} + b_t\Delta x_{t+1},$$

where $x_{t+1}\Delta b_{t+1}$ measures the effect of the change in b_t given the value of x_{t+1}, while $b_t\Delta x_{t+1}$ measures the effect of the change in x_t, given b_t. Thus, the MC condition $|x_{t+1}\Delta b_{t+1}| \leq |b_t\Delta x_{t+1}|$ in (6) implies that the first term $x_{t+1}\Delta b_{t+1}$ cannot change the sign of $\Delta E(d_t|\mathbf{x}_t)$. Therefore, as $b_t > 0$, $\Delta(b_{t+1}x_{t+1})$ has the same sign as Δx_{t+1}, such that $\Delta E(d_t|\mathbf{x}_t)\Delta x_{t+1} \geq 0$. ■

Proof of Lemma 3. Positive homogeneity follows from

$$\Delta(\lambda z_{t+1})\Delta x_{t+1} = \lambda\Delta z_{t+1}\Delta x_{t+1} \geq 0.$$

Next, additivity holds by

$$\Delta(z_{t+1} + y_{t+1})\Delta x_{t+1} = \Delta z_{t+1}\Delta x_{t+1} + \Delta y_{t+1}\Delta x_{t+1} \geq 0,$$

and finally multiplicativity follows from

$$\Delta(y_{t+1}z_{t+1})\Delta x_{t+1} = z_{t+1}\Delta y_{t+1}\Delta x_{t+1} + y_t\Delta z_{t+1}\Delta x_{t+1} \geq 0.$$

■

Proof of Theorem 1. From Assumption 4 applied to (17), it is seen that for $n \to \infty$,

$$b_t x_t (\sum_{k=1}^{n} \gamma^k [L, U]^k) + \gamma^n Q E_t^{(n-1)}(\mathcal{I}_{t+n}^p) \to b_t x_t \sum_{k=1}^{\infty} \gamma^k [L_\gamma, U_\gamma].$$

■

Proof of Theorem 2. To prove the result we consider the case where $\theta_t/b_t = U_\gamma$ is as large as possible, and $\theta_{t+1}/b_{t+1} = L_\gamma$ as small as possible, given the constraints that θ_t/b_t and $\theta_{t+1}/b_{t+1} \in [L_\gamma, U_\gamma]$. We find from the moderate condition (8) that because $\Delta x_{t+1} \geq 0$

$$b_{t+1}x_{t+1} \leq b_t(x_{t+1} + |x_{t+1} - x_t|) = b_t(x_t + 2\Delta x_{t+1}).$$

We then evaluate the difference

$$\begin{aligned}\Delta p_{t+1} &= \theta_{t+1}x_{t+1} - \theta_t x_t = b_{t+1}x_{t+1}L_\gamma - b_t x_t U_\gamma \leq b_t(x_t + 2\Delta x_{t+1})L_\gamma - b_t x_t U_\gamma \\ &= L_\gamma b_t x_t[1 + 2\frac{\Delta x_{t+1}}{x_t} - \frac{U_\gamma}{L_\gamma}] \\ &< L_\gamma b_t x_t[1 + \frac{U-L}{L} - \frac{U_\gamma}{L_\gamma}] = L_\gamma b_t x_t[\frac{U}{L} - \frac{U_\gamma}{L_\gamma}] < 0.\end{aligned}$$

■

Proof of Theorem 3. If $\Delta x_{t+1} \geq 0$ and $\Delta(\theta_{t+1}/b_{t+1}) \geq 0$, then $\theta_{t+1} \geq \theta_t b_{t+1}/b_t$ and therefore

$$\Delta p_{t+1} = \theta_{t+1}x_{t+1} - \theta_t x_t \geq \theta_t b_{t+1}x_{t+1}/b_t - \theta_t x_t = \theta_t \Delta(b_{t+1}x_{t+1})/b_t,$$

which is positive because $b_t x_t$ co-moves positively with x_t, such that $\Delta(b_{t+1}x_{t+1}) \geq 0$.

If $\Delta x_{t+1} \leq 0$ and $\Delta(\theta_{t+1}/b_{t+1}) \leq 0$, then $\theta_{t+1} \leq \theta_t b_{t+1}/b_t$ and therefore

$$\Delta p_{t+1} = \theta_{t+1}x_{t+1} - \theta_t x_t \leq \theta_t b_{t+1}x_{t+1}/b_t - \theta_t x_t = \theta_t \Delta(b_{t+1}x_{t+1})/b_t,$$

which is negative because $b_t x_t$ co-moves positively with x_t, such that $\Delta(b_{t+1}x_{t+1}) \leq 0$. ■

Proof of Corollary 4. The proof is the same as for Lemma 1. ■

B The Lower and Upper Bounds

We derive expressions for the coefficients L and U, see (11)

Assumption 1 *Assume that*

$$x_t = \exp(\sum_{i=1}^{t} \varepsilon_{xi} - t\sigma^2/2) = \prod_{i=1}^{t} \exp(\varepsilon_{xi} - \sigma^2/2), \tag{44}$$

where ε_{xt} i.i.d.$N(0, \sigma^2)$.

We first prove a general result about the Gaussian distribution and then formulate the results in a Corollary for the coefficients we have defined.

Lemma 4 *Let $\varphi(\cdot)$ and $\Phi(\cdot)$ be the density and distribution function of the standard Gaussian distribution. Then for any $a, b \in \mathbb{R}$*

$$\int_a^b \exp(x - \frac{1}{2}\sigma^2)\varphi(x/\sigma)dx/\sigma = \Phi(b/\sigma - \sigma) - \Phi(a/\sigma - \sigma).$$

Proof of Lemma 4. By definition of $\varphi(\cdot)$, standard manipulations imply that

$$\int_a^b \exp(x - \frac{1}{2}\sigma^2)\varphi(\frac{x}{\sigma})\frac{dx}{\sigma} = \frac{1}{\sqrt{2\pi}}\int_a^b \exp(-\frac{1}{2\sigma^2}(x-\sigma^2)^2)\frac{dx}{\sigma}.$$

Substituting $u = (x - \sigma^2)/\sigma$, we find that

$$\int_{a/\sigma-\sigma}^{b/\sigma-\sigma} \varphi(u)du = \Phi(b/\sigma - \sigma) - \Phi(a/\sigma - \sigma).$$

■

The coefficients U and L are defined in (11). We find from

$$x_{t+1} = x_t \exp(\varepsilon_{xt+1} + \mu),$$

that

$$E_t(x_{t+1} \pm |\Delta x_{t+1}|)^+ = x_t E\{\exp(\varepsilon_{xt+1} + \mu) \pm |\exp(\varepsilon_{xt+1} + \mu) - 1|\}^+.$$

and therefore calculate

$$L = E\{\exp(\varepsilon_{xt+1} + \mu) - |\exp(\varepsilon_{xt+1} + \mu) - 1|\}^+,$$
$$U = E\left\{\exp(\varepsilon_{xt+1} + \mu) + |\exp(\varepsilon_{xt+1} + \mu) - 1|\right\}.$$

Lemma 5 *If Assumption 1 holds for x_t, then it follows from Lemma 4 that the following relations hold*

$$x_t \exp(\varepsilon_{x,t+1} - \sigma^2/2) = x_{t+1}, \tag{45}$$
$$E\exp\left(\varepsilon_{xt} - \sigma^2/2\right) = 1, \tag{46}$$

$$L = 1 - \Phi(\frac{\sigma}{2} - \frac{\log 2}{\sigma}) + 2\{\Phi(-\frac{\sigma}{2}) - \Phi(-\frac{\log 2}{\sigma} - \frac{\sigma}{2})\}, \tag{47}$$
$$U = 1 + 2\{\Phi(\frac{\sigma}{2}) - \Phi(-\frac{\sigma}{2})\}. \tag{48}$$

Proof of Lemma 5. We find (45) from (44), and using Lemma 4, for $b = \infty$ and $a = -\infty$, we find (46).

$$E\exp(\varepsilon_t - \sigma^2/2) = \{\Phi(\infty/\sigma - \sigma) - \Phi(-\infty/\sigma - \sigma)\} = 1.$$

To prove (47) note that by definition, for $\mu = -\sigma^2/2$,

$$\{\exp(\varepsilon_{xt} + \mu) - |\exp(\varepsilon_{xt} + \mu) - 1|\}^+ = \begin{cases} 1 & \varepsilon_{xt} + \mu > 0 \\ 2\exp(\varepsilon_{xt} + \mu) - 1 & -\log 2 < \varepsilon_{xt} + \mu < 0 \end{cases}$$

We define the sets $A = \{-\log 2 < \varepsilon_{xt} + \mu < 0\}$ and $B = \{\varepsilon_{xt} + \mu > 0\}$ and find that the probabilities $P(A) = \Phi(\sigma/2) - \Phi(\sigma/2 - \log 2/\sigma)$ and $P(B) = 1 - \Phi(\sigma/2)$, such that

$$\begin{aligned} L &= P(B) - P(A) + 2E\left(\exp(\varepsilon_{xt} + \mu)\mathbf{1}_A\right) \\ &= 1 - \Phi(\sigma/2) + \Phi(\sigma/2) - \Phi(\sigma/2 - \log 2/\sigma) + 2\int_{-\log 2 - \mu}^{-\mu} \exp(x + \mu)\varphi(x/\sigma)dx/\sigma. \end{aligned}$$

Setting $\mu = -\sigma^2/2$, we find

$$\int_{-\log 2 + \sigma^2/2}^{\sigma^2/2} \exp(x - \sigma^2/2)\varphi(x/\sigma)dx/\sigma = \{\Phi(\sigma/2 - \sigma) - \Phi(-\log 2/\sigma - \sigma/2)\},$$

and by collecting terms we find

$$L = 1 - \Phi(\sigma/2 - \log 2/\sigma) + 2\{\Phi(\sigma/2 - \sigma) - \Phi(-\log 2/\sigma - \sigma/2)\}.$$

Finally, we want to prove (48). We find

$$\exp(\varepsilon_{xt} + \mu) + |\exp(\varepsilon_{xt} + \mu) - 1| = \begin{cases} 2\exp(\varepsilon_{xt} + \mu) - 1 & \varepsilon_{xt} + \mu > 0 \\ 1 & \varepsilon_{xt} + \mu < 0, \end{cases}$$

such that for $\mu = -\sigma^2/2$

$$U = 2\int_{\sigma^2/2}^{\infty} \exp(x - \sigma^2/2)\varphi(x/\sigma)dx/\sigma - (1 - \Phi(\sigma/2)) + \Phi(\sigma/2).$$

By Lemma 4 for $b = \infty$ and $a = \sigma^2/2$ this can be reduced to

$$U = 2(1 - \Phi(-\sigma/2)) - (1 - 2\Phi(\sigma/2)) = 1 + 2(\Phi(\sigma/2) - \Phi(-\sigma/2))$$

$\blacksquare$

Lemma 6 *Under Assumption 1 for $\mu = -\sigma^2/2$,*

$$1 + 2(\Phi(\sigma/2) - \Phi(-\sigma/2) = 1 + \sigma\sqrt{\frac{2}{\pi}} + O\left(\sigma^2\right).$$

Proof of Lemma 6. The proof follows by a Taylor's expansion around $\sigma = 0$. ∎

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